

Conic Sections in LibreCAD

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1 The nearest point on an ellipse to a given point

An ellipse in the coordinates orientated along its major and minor axes is given as,

$$\begin{cases} x = a \cos t \\ y = b \sin t \end{cases} \quad 0 \leq t < 2\pi \quad (1)$$

the distance to a given point (x, y) ,

$$\begin{aligned} s^2 &= (x - a \cos t)^2 + (y - b \sin t)^2 \\ &= x^2 + y^2 + a^2 \cos^2 t + b^2 \sin^2 t - 2xa \cos t - 2yb \sin t \end{aligned} \quad (2)$$

The stationary points can be found by the zero points of its first order derivative,

$$\frac{d(s^2)}{dt} = -a^2 \sin 2t + b^2 \sin 2t + 2xa \sin t - 2yb \cos t = 0 \quad (3)$$

$$\begin{aligned} \sin t(2xa + 2(b^2 - a^2) \cos t) &= 2yb \cos t \\ \sin t &= \frac{2yb \cos t}{2xa + 2(b^2 - a^2) \cos t} \end{aligned} \quad (4)$$

The second order derivative is,

$$\frac{d^2(s^2)}{dt^2} = 2xa \cos t + 2yb \sin t + 2(b^2 - a^2) \cos 2t$$

To solve the equation of stationary point, let $u = \cos t$, Eq. (4) is equivalent to a quartic equation of u .

$$\begin{aligned}
u &= \cos t \\
(1 - u^2)(2xa + 2(b^2 - a^2)u)^2 &= 4y^2b^2u^2 \\
(1 - u^2)(\gamma u - \alpha)^2 &= \beta^2u^2 \\
\alpha &= 2ax \\
\beta &= 2by \\
\gamma &= 2(a^2 - b^2) \\
(u^2 - 1)(\gamma u - \alpha)^2 + \beta^2u^2 &= 0 \\
\gamma^2u^4 - 2\alpha\gamma u^3 + (\alpha^2 + \beta^2 - \gamma^2)u^2 + 2\alpha\gamma u - \alpha^2 &= 0
\end{aligned}$$

The minimum points have positive second order derivatives,

$$\frac{d^2(s^2)}{dt^2} = -2(a^2 - b^2) \cos 2t + 2ax \cos t + 2by \sin t > 0 \quad (5)$$

$$\gamma(1 - 2u^2) + \alpha u + \frac{\beta^2 u}{\alpha - \gamma u} > 0$$

2 Ellipse-Ellipse Intersection

Choose the coordinate axes to be along the major and minor axes of one of the two ellipses, the equations for the two ellipses have the following form,

$$\begin{cases} A_{000}x^2 + A_{011}y^2 - 1 = 0 \\ A_{100}x^2 + 2A_{101}xy + A_{111}y^2 + B_{10}x + B_{11}y - 1 = 0 \end{cases} \quad (6)$$

Regroup the ellipse equations as quadratic equations of x ,

$$\begin{cases} \alpha_2x^2 + \alpha_0 = 0 \\ \beta_2x^2 + \beta_1x + \beta_0 = 0 \end{cases} \quad (7)$$

With the coefficients as polynomials of y ,

$$\begin{cases} \alpha_2 &= A_{000} \\ \alpha_0 &= A_{011}y^2 - 1 \\ \beta_2 &= A_{111} \\ \beta_1 &= 2A_{101}y + B_{10} \\ \beta_0 &= A_{111}y^2 + B_{11}y - 1 \end{cases} \quad (8)$$

Eliminate x from Eqs. (7), the Bézout determinant has the form,

$$(\alpha_2\beta_0 - \alpha_0\beta_2)^2 + \alpha_2\alpha_0\beta_1^2 = 0 \quad (9)$$

Eq. (9) is a quartic equation of y ,

$$u_4 y^4 + u_3 y^3 + u_2 y^2 + u_1 y + u_0 = 0 \quad (10)$$

where the parameters u_i are,

$$\begin{cases} u_4 &= v_1^2 + 2A_{101}A_{011}v_0 \\ u_3 &= v_0v_8 + 2v_3v_1 \\ u_2 &= 2(v_1v_4 - A_{101}v_0) + v_3^2 + v_2v_8/2 \\ u_1 &= 2(v_3v_4 - B_{10}v_0) \\ u_0 &= v_4^2 - B_{10}v_2 \end{cases} \quad (11)$$

with the parameters v_i ,

$$\begin{cases} v_0 &= 2A_{000}A_{100} \\ v_1 &= A_{000}A_{111} - A_{100}A_{011} \\ v_2 &= A_{000}B_{10} \\ v_3 &= A_{000}B_{11} \\ v_4 &= A_{000}C_1 + A_{100} \\ v_8 &= 2A_{011}B_{10} \end{cases} \quad (12)$$

$$\begin{aligned} \alpha_0 &= A_{000} \\ \gamma_0 &= A_{011}y^2 - 1 \\ \alpha_1 &= A_{100} \\ \beta_1 &= 2A_{101}y + B_{10} \\ \gamma_1 &= A_{111}y^2 + B_{11}y - 1 \end{aligned}$$

$$\alpha_2 = A_{000}$$

$$\alpha_0 = A_{011}y^2 - 1$$

$$\begin{aligned} \beta_2 &= A_{100} \\ \beta_1 &= 2A_{101}y + B_{10} \\ \beta_0 &= A_{111}y^2 + B_{11}y + C_1 \end{aligned}$$

$$\begin{aligned} \alpha_2 x^2 + \alpha_0 &= 0 \\ \beta_2 x^2 + \beta_1 x + \beta_0 &= 0 \end{aligned}$$

$$\begin{aligned}\alpha_2\beta_1x + \alpha_2\beta_0 - \alpha_0\beta_2 &= 0 \\ (\alpha_2\beta_0 - \alpha_0\beta_2)^2 + \alpha_0\alpha_2\beta_1^2 &= 0\end{aligned}$$

$$x = \frac{\alpha_0\beta_2 - \alpha_2\beta_0}{\alpha_2\beta_1}$$

Equations,

$$\begin{aligned}A_{000}x^2 + A_{011}y^2 - 1 &= 0 \\ A_{100}x^2 + (2A_{101}y + B_{10})x + (A_{111}y^2 + B_{11}y + C_1) &= 0\end{aligned}$$

Elementing x^2, x ,

$$A_{000}[(2A_{101}y + B_{10})x + (A_{111}y^2 + B_{11}y + C_1)] - A_{100}(A_{011}y^2 - 1)$$

$$[2A_{000}A_{101}y + A_{000}B_{10}]x + (A_{000}A_{111} - A_{100}A_{011})y^2 + A_{000}B_{11}y + A_{000}C_1 + A_{100} = 0$$

$$x = -\frac{(A_{000}A_{111} - A_{100}A_{011})y^2 + A_{000}B_{11}y + A_{000}C_1 + A_{100}}{2A_{000}A_{101}y + A_{000}B_{10}}$$

$$[A_{000}(A_{111}y^2 + B_{11}y + C_1) - (A_{011}y^2 - 1)A_{100}]^2 + (A_{011}y^2 - 1)A_{000}(2A_{101}y + B_{10})^2$$

$$y^4 : A_{000}^2A_{111}^2 - 2A_{000}A_{011}A_{100}A_{111} + 4A_{000}A_{011}A_{101}^2 + A_{011}^2A_{100}^2$$

$$y^3 : 2A_{000}^2A_{111}B_{11} - 2A_{000}A_{011}A_{100}B_{11} + 4A_{000}A_{011}A_{101}B_{10}$$

$$\begin{aligned}y^2 : 2A_{000}^2A_{111}C_1 - 2A_{000}A_{011}A_{100}C_1 + A_{000}^2B_{11}^2 + A_{000}A_{011}B_{10}^2 + 2A_{000}A_{100}A_{111} - 4A_{000}A_{101}^2 - 2A_{011}A_{100}^2 \\ 2(A_{000}C_1 + A_{100})(A_{000}A_{111} - A_{011}A_{100}) + A_{000}^2B_{11}^2 + A_{000}A_{011}B_{10}^2 - 4A_{000}A_{101}^2\end{aligned}$$

$$y : 2A_{000}^2B_{11}C_1 + 2A_{000}A_{100}B_{11} - 4A_{000}A_{101}B_{10}$$

$$y^0 : A_{000}^2C_1^2 + 2A_{000}A_{100}C_1 - A_{000}B_{10}^2 + A_{100}^2$$

3 intersection between an ellipse and a line

$$\begin{aligned}x &= x_0 + k_x t \\y &= y_0 + k_y t\end{aligned}$$

$$\begin{aligned}\frac{(x_0 + k_x t)^2}{a^2} + \frac{(y_0 + k_y t)^2}{b^2} &= 1 \\(\frac{k_x^2}{a^2} + \frac{k_y^2}{b^2})t^2 + 2(\frac{x_0 k_x}{a^2} + \frac{y_0 k_y}{b^2})t + (\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - 1) &= 0\end{aligned}$$

4 Scale an ellipse

$$\begin{aligned}x &= a \cos t \\y &= b \sin t\end{aligned}$$

rotated to angle θ ,

$$\begin{aligned}x &= a \cos t \cos \theta - b \sin t \sin \theta \\y &= a \cos t \sin \theta + b \sin t \cos \theta\end{aligned}$$

scale x and y

$$\begin{aligned}x &= k_x (a \cos t \cos \theta - b \sin t \sin \theta) \\y &= k_y (a \cos t \sin \theta + b \sin t \cos \theta)\end{aligned}$$

The distance to ellipse center,

$$\begin{aligned}r^2 &= x^2 + y^2 \\&= a^2(k_x^2 \cos^2 \theta + k_y^2 \sin^2 \theta) \cos^2 t + b^2(k_x^2 \sin^2 \theta + k_y^2 \cos^2 \theta) \sin^2 t + ab \sin \theta \cos \theta (k_y^2 - k_x^2) \sin 2t\end{aligned}$$

Define,

$$\begin{aligned}A &= \frac{1}{2}a^2(k_x^2 \cos^2 \theta + k_y^2 \sin^2 \theta) \\B &= \frac{1}{2}b^2(k_x^2 \sin^2 \theta + k_y^2 \cos^2 \theta) \\C &= ab \sin \theta \cos \theta (k_y^2 - k_x^2)\end{aligned}$$

Then,

$$r^2 = A + B + (A - B) \cos 2t + C \sin 2t$$

The major radius is the maximum of r ,

$$r_{max} = A + B + \sqrt{(A - B)^2 + C^2} \quad (13)$$

the minor radius is the minimum of r ,

$$r_{min} = A + B - \sqrt{(A - B)^2 + C^2} \quad (14)$$

and the orientaion of major axis is corresponding to ellipse angle t of the unscaled ellipse,

$$\tan t = \frac{C}{A - B} \quad (15)$$

5 Tangent of an ellipse

the tangent at a given point has the slope of,

$$\frac{dy}{dx} = -\frac{b \cos \theta}{a \sin \theta}$$

When tangent vector $(-a \sin \theta, b \cos \theta)$ is normal to a given direction (x, y)

$$ax \sin \theta = by \cos \theta$$

$$\tan \theta = \frac{b}{a} \frac{y}{x}$$

6 Ellipse from 4 points with axes in $x - /y$ -directions

The ellipse is determined by a linear equation set of the form,

$$ax^2 + bxy + cy^2 + dx + ey = 1$$

7 Circle from three points

$$\begin{cases} |\vec{r}_a - \vec{r}|^2 &= |\vec{r}|^2 \\ |\vec{r}_b - \vec{r}|^2 &= |\vec{r}|^2 \end{cases}$$

$$\begin{cases} \vec{r}_a \cdot \vec{r} &= |\vec{r}_a|^2/2 \\ \vec{r}_b \cdot \vec{r} &= |\vec{r}_b|^2/2 \end{cases}$$

8 Ellipse by foci and a point on ellipse

Ellipse is defined as having a constant total distance to the two foci

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2d$$

$$(x^2 + y^2 + c^2) + \sqrt{(x^2 + y^2 + c^2)^2 - 4x^2c^2} = 2d^2$$

$$\frac{x^2}{d^2} + \frac{y^2}{d^2 - c^2} = 1$$

9 Ellipse arc length

The arc length over angle range,

$$\begin{aligned} s &= \int_0^\theta \sqrt{dx^2 + dy^2} \\ &= \int_0^\theta \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} dt \\ &= a \int_0^\theta \sqrt{1 - \frac{a^2 - b^2}{a^2} \cos^2 t} dt \\ &= a \int_0^\theta \sqrt{1 - k^2 \sin^2 t} dt \\ &= a[E(k) - E(\theta, k)] \end{aligned}$$

first order derivative,

$$\begin{aligned} \frac{ds}{d\theta} &= a \sqrt{1 - k^2 \cos^2 t} \\ \frac{d\theta}{ds} &= \frac{1}{a \sqrt{1 - k^2 \cos^2 t}} \\ ds &= \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} dt \end{aligned}$$

the curvature of an ellipse,

$$\begin{aligned} \kappa &= \frac{ab}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}^3} \\ ds &= \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} dt \end{aligned}$$

$$d\theta = \kappa ds = \frac{ab}{a^2 \sin^2 t + b^2 \cos^2 t} dt$$

$$dt = \frac{a^2 \sin^2 t + b^2 \cos^2 t}{ab} d\theta$$

second order derivative

$$\frac{d^2 s}{d\theta^2} = a \frac{k^2 \sin t \cos t}{\sqrt{1 - k^2 \cos^2 t}}$$

where $k = \sqrt{1 - b^2/a^2}$

10 Ellipse centered at origin and passing three points

An ellipse centered at origin has the form,

$$ax^2 + 2bxy + cy^2 = 1$$

the parameters are determined by a linear equation set, and the ellipse axes are recovered by finding the eigen vectors and eigen values of the quadratic form

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

11 Problem of Appollonius

The statement of the problem: to construct a common tangent circle of three given circles.

Algebraic solution of the Appollonius' Problem. The three given circles have their centers and radii as, (\vec{C}_1, R_1) , (\vec{C}_2, R_2) and (\vec{C}_3, R_3) , respectively. The common tangent circle has its center and radius as (\vec{C}_0, R_0) to be determined. The tangent conditions:

$$\begin{aligned} |\vec{C}_0 - \vec{C}_1|^2 &= (R_0 + R_1)^2 \\ |\vec{C}_0 - \vec{C}_2|^2 &= (R_0 + R_2)^2 \\ |\vec{C}_0 - \vec{C}_3|^2 &= (R_0 + R_3)^2 \end{aligned} \tag{16}$$

where R_i $i = 1, 2, 3$ are either positive or negative to give a total of 8 possible solutions.

For Eqs. (16), subtract the last equation from the first two to get rid of quadratic terms,

$$\begin{cases} 2(\vec{C}_3 - \vec{C}_1) \cdot \vec{C}_0 &= |\vec{C}_3|^2 - |\vec{C}_1|^2 + R_1^2 - R_3^2 + 2(R_1 - R_3)R_0 \\ 2(\vec{C}_3 - \vec{C}_2) \cdot \vec{C}_0 &= |\vec{C}_3|^2 - |\vec{C}_2|^2 + R_2^2 - R_3^2 + 2(R_2 - R_3)R_0 \end{cases} \quad (17)$$

Solve Eqs. (17) for \vec{C}_0 as a linear function of R_0 ,

$$\begin{cases} x_0 &= P_{x1} + Q_{x1}R_0 \\ y_0 &= P_{y1} + Q_{y1}R_0 \end{cases} \quad (18)$$

In combination with Eq. (16),

$$|\vec{P} + \vec{Q}R_0 - \vec{C}_0|^2 = (R_0 + R_1)^2$$

$$(|\vec{Q}|^2 - 1)R_0^2 + 2[(\vec{P} - \vec{C}_0) \cdot \vec{Q} - R_1]R_0 + |\vec{C}_0 - \vec{P}|^2 - R_1^2 = 0$$